On the self-CPG curves and the Björling problem

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Abstract

Schwarz's solution to the Björling problem leads to an equivalence class of spatial strips $S(t) = (c(t), n(t))$ which produce equivalent minimal surfaces. For the particular case when the generating strip $S(t)$ belongs to some plane E and $c(t)$ is a symmetric curve with respect to some straight line in E, the symmetries of the minimal surface permit us to identify another planar (geodesic) curve $\tilde{c}(t)$ that we call the CPG curve to $c(t)$. A simple symmetric argument shows that *self-CPG* curves produce minimal surfaces whose adjoint surface contains another *self-CPG* curve. We ask for minimal surfaces with *self-CPG* curves which are *self-adjoints*.

1 Introduction

Schwarz's solution to the Björling problem permit us to construct a lot of minimal surfaces from real analytic strips $S(t) = (c(t), n(t))$, where $c : I \to \mathbb{R}^3$ is a real analytic curve and $n : \mathbb{R}^3 \to \mathbb{R}^3$ is an unitary vector field over $c(t)$ such that $\langle \dot{c}(t), n(t) \rangle \equiv 0$. For the case when $S(t)$ is contained in some plane E, the unitary vector field $n(t)$ is recovered from the principal normal field $\mathfrak{n}(t) = \ddot{c}(t)/\|\ddot{c}(t)\|$ assuming that $c(t)$ is parameterized by arc lenght. In this situation, $c(t)$ is a plane geodesic of the minimal surface $X : \Omega \to \mathbb{R}^3$ which solves the Björling problem.

In a general context, we can consider the set of viable strips $\mathscr{S} = \{S(t) = (c(t), n(t))\}$ (see section 2.2) and consider equivalence classes $[S(t)]$ such that for every $\tilde{S}(t) \in [S(t)]$ the minimal surface $\tilde{X}(w)$ which solves the Björling problem is congruent to $X(w)$. The space $\mathscr{S}$ is very big, however we are interested in a particular class of strips, the planar strips which posses a simple symmetry. Suppose that the planar curve $c(t)$ has a line of symmetry $\mathscr{L}$ which intersects it perpendicularly. A simple analysis of the symmetries shows that X will have another symmetry plane $E_{\mathscr{L}}$ which intersects E perpendicularly along $\mathscr{L}$. The plane $E_{\mathscr{L}}$ will contain another planar geodesic $\tilde{c}(t) \subset X$. We say that $\tilde{c}(t)$ is the *conjugated perpendicular geodesic* (CPG) to $c(t)$. Evidently, both belongs to the same equivalence class $[S(t)]$ for $S(t) = (c(t), \mathfrak{n}(t))$.

In this paper we are concerned with minimal surfaces which are solutions to the Björling problem for strips $S(t)$ whose supporting curves $c(t)$ are the *CPG* of themselves, up to an specific rotation. We call them *self-CPG* curves. We give examples of self-*CPG* curves which comes from some classical minimal surfaces and we relate the self-*CPG* condition with the *self-adjoint* property of minimal surfaces.

2 The Björling equivalence for planar curves

First we recall some well-known facts from the theory of minimal surfaces. We follow the description given by Dierkes *et al.* in [2].

2.1 Parametric minimal surfaces and geodesics

Let $\tilde{\Omega}$ be an open simply connected subset of $\mathbb{R}^2$ and let $X : \tilde{\Omega} \to \mathbb{R}^3$ be a mapping of class at least C^2 which sends $w = (u, v) \in \tilde{\Omega}$ to $X(u, v) \in \mathbb{R}^3$. The image of X in $\mathbb{R}^3$ is a minimal surface if the mapping X satisfies the equations

$$\Delta X = 0 \tag{1}$$

$$|X_u|^2 = |X_v|^2, \qquad \langle X_u, X_v \rangle = 0 \tag{2}$$

on $\tilde{\Omega}$, where Δ is the Laplace-Beltrami operator. In the rest of this document we identify the mapping with its image and we say that X is a minimal surface in $\mathbb{R}^3$.

We define the *adjoint surface* to X on $\tilde{\Omega}$ as the surface X^* which solves the *Cauchy-Riemann equations*

$$X_u = X_v^*, \qquad X_v = -X_u^*, \tag{3}$$

from where we obtain that the adjoint surface X^* to a minimal surface X is also a minimal surface. This fact permit us to state the problem from the complex point of view identifying $\mathbb{C} \cong \mathbb{R}^2$.

Let $f : \Omega \to \mathbb{C}^3$ be a holomorphic mapping defined on the open domain $\Omega = \tilde{\Omega} \setminus \{Sing(f)\}$, lets denote by $f'(w) = \frac{\partial f(w)}{\partial w}$ the derivative of $f(w)$ with respect to w, and by $\langle , \rangle : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ the Hermitian inner product on $\mathbb{C}^3$. If $\langle f'(w), f'(w) \rangle \equiv 0$, vanish identically on Ω, the map $f(w)$ is called an *isotropic* (complex) curve, and the real and imaginary components

$$X(w) := \Re(f(w)) \quad \text{and} \quad X^*(w) := \Im(f(w)), \tag{4}$$

define minimal surfaces in $\mathbb{R}^3$, whether or not Ω is simply connected.

The tangent space at any regular point $w \in \Omega$ is spanned by the vectors X_u and X_v. Additionally, at any $w \in \Omega$, the exterior product $X_u \wedge X_v$ does not vanish and we identify this bivector with its normal (perpendicular) in $\mathbb{R}^3$ in the traditional way. In a neighborhood of w the *unitary normal vector* to X is well defined and it is given by

$$N = \frac{X_u \wedge X_v}{\|X_u \wedge X_v\|}. \tag{5}$$

The map $N : \Omega \to \mathbb{S}^2$ corresponds to the composition $N(w) := N \circ X(w)$ and it is called the Gauss map. Since the image of any subset $C \subset \Omega$ in the domain of N belongs to $\mathbb{S}^2$ then $N(C)$ is known as the spherical image of $X(C)$.

Two minimal surfaces $\hat{X}$ and X are said *congruents* if there exist an isometry φ and a real number $\alpha \in \mathbb{R}_*$ such that $\hat{X} = \alpha\varphi(X)$, where $\mathbb{R}_*$ is the real multiplicative group. If $\alpha = 1$, we call them *equivalent* surfaces.

In the rest of the section the curves are parametrized by arc lenght. For any regular curve $c : I \to \mathbb{R}^3$ we call *tangent vector* to $\mathbf{t}(t) = \dot{c}(t)$ which is a unitary vector, $\kappa(t) = \|\dot{\mathbf{t}}(t)\|$ is its *curvature*, $\mathfrak{n} = \dot{\mathbf{t}}(t)/\kappa(t)$ its *principal normal* and $\mathfrak{b}(t) = \mathfrak{n}(t) \times \mathbf{t}(t)$ its *binormal*. This give us an orthonormal frame $\mathscr{F} = \{\mathbf{t}, \mathfrak{b}, \mathfrak{n}\}$ over $c(t)$ from the intrinsic geometry of the curve.

Now, we consider the curve $\gamma : I \to \Omega$ such that $c(t) := X \circ \gamma$ is parameterized by arc lenght. We define the *normal* by $\mathbf{n}(t) := N(\hat{c}(t))$ and the *side normal* by $\mathbf{s}(t) := \mathbf{n}(t) \times \mathbf{t}(t)$. We obtain another orthonormal frame $\hat{\mathscr{F}} = \{\mathbf{t}, \mathbf{s}, \mathbf{n}\}$ over $c(t)$ from the intrinsic geometry of X. Both frames are related by

$$\begin{aligned} \cos\theta(t) &= \langle \mathbf{n}(t), \mathfrak{n}(t) \rangle, \\ &= \langle \mathbf{s}(t), \mathfrak{b}(t) \rangle. \end{aligned}$$

Since $\mathbf{t}(t)$ is an unitary vector then $\mathbf{n}(t)$ is a linear combination

$$\mathbf{n}(t) = \sin\theta(t)\mathfrak{b}(t) + \cos\theta(t)\mathfrak{n}(t).$$

We define by $\kappa_g(t) = \kappa(t)\sin\theta(t)$ the *geodesic curvature* and by $\kappa_n(t) = \kappa(t)\cos\theta(t)$ the *normal curvature* of $c(t) \subset X(w)$ for the parameter t.

A curve $c \subset X$ is called a *geodesic* of X if its geodesic curvature $\kappa_g(t)$ vanishes for all $t \in I$, it is called an *asymptotic* curve of X if its normal curvature $\kappa_n(t)$ vanishes everywhere and it is called a *line of curvature* if $\dot{c}(t)$ is proportional to a principal direction of X along $c(t)$, whether or not $c(t)$ is parametrized by arc lenght.

2.2 The Björling's problem

Let $c : I \to \mathbb{R}^3$ be a real analytic curve which admits an holomorphic extension $c(w) \subset \mathbb{C}^3$ and such that $\dot{c}(t) \neq 0$ almost everywhere. Over the curve $c(t)$, consider a non-vanishing unitary vector field $n : \mathbb{R}^3 \to \mathbb{S}^2$ perpendicular to the tanget vector $\mathbf{t}(t) = \dot{c}(t)$, *i.e.* $\langle \mathbf{t}(t), n(t) \rangle \equiv 0$. The couple $S(t) = (c(t), n(t))$ defines a real analytic strip in $\mathbb{R}^3$.

Given a strip $S(t)$ as before, the Björling's problem concerns in to find a minimal surface $X : \Omega \to \mathbb{R}^3$ whose normal field $N : \Omega \to \mathbb{S}^2$ contains the strip $S(t)$. It means that $c(t)$ must belongs to $X(w)$ fullfiling the following properties

$$X(t) = c(t), \qquad \forall t \in I \subset \Omega, \tag{6}$$

$$N(t) = n(t), \qquad \forall t \in I \subset \Omega. \tag{7}$$

It is immediate from conditions (7) and $\langle \mathbf{t}(t), n(t) \rangle \equiv 0$ that $c(t)$ is a geodesic in $X(w)$.

Schwarz has proposed a solution in [8] (reproduced in [9]) using the Weierstrass representation which was generalized by the Cauchy-Kovalevskaya theorem. Schwarz's solution to Björling's problem is given by

$$X(w) = \Re\left(c(w) - i\int_{w_0}^{w} n(z)\wedge c'(z)dz\right), \qquad z, w \in \Omega \subset \mathbb{C}. \tag{8}$$

where $c'(w) = dc(w)/dw$.

We say that $S(t) = (c(t), n(t))$ are the *Björling data* for X. Ω is associated to $S(t)$ as the maximal domain for the holomorphic extension and, in general, they are open domains on Riemann surfaces. We say that a strip $S(t)$ is *viable* if there exists a regular parameterization of c whose holomorphic extension is defined over a punctured Riemann surface. In particular, all the algebraic curves gives viables strips.

The space of viable strips $\mathscr{S} = \{S(t) = (c(t), n(t)) | S(t) \text{ is viable}\}$ permit us to consider local and global parameterized curves as the same Björling data. Consequently the "space" of complete minimal surfaces in the Euclidian space $\mathscr{X} = \left\{X \subset \mathbb{R}^3 | X \text{ is a minimal surface}\right\}$ will consider small open subsets from a minimal surface and the minimal surface itself as the same object. We didn't have studied the implications of this consideration on the *Schwarzian chain problem*.

We define the *Björling transformation* of a strip $S(t)$ as the application

$$\mathfrak{B} : \mathscr{S} \rightarrow \mathscr{X} \tag{9}$$

$$S(t) \mapsto \Re\left(c(w) - i\int_{w_0}^{w} n(z)\wedge c'(z)dz\right). \tag{10}$$

which sends the strip $S(t)$ to the minimal surface $X(w)$.

We can give a simplified strip $S(t)$ when the curve $c(t)$ has particular properties. A classical result of O. Bonnet [1] says that it is possible to determine X when the curve c belongs to X in the following cases: a) c is a geodesic, b) c is an asymptotic line, c) c is a line of curvature, d) c is a shadow line, e) c is a perspective line. Then consider a planar curve $c : I \to \mathbb{R}^3$ contained in the plane E and the orthonormal intrinsic frame $\{\mathfrak{t}, \mathfrak{b}, \mathfrak{n}\}$ over $c(t)$. Since $c(t)$ is a planar curve then the binormal vector $\mathfrak{b}$ coincides with the normal e to E over $c(t)$. Define the normal $n(t)$ over $c(t)$ by

$$n(t) = \mathfrak{b}(t)\cos\varphi(t) + \mathfrak{n}(t)\sin\varphi(t), \qquad \varphi(t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{11}$$

where $\mathfrak{t}(t) = \dot{c}(t)/\|\dot{c}(t)\|$. This gives the condition $\langle n(t), \mathfrak{b}(t)\rangle \equiv \cos\varphi$, for all $t \in I$. We obtain an analytic strip $S(t)$ whose Björling transformation is

$$\mathfrak{B}(S) = \Re\left\{c(w) - i\left(\cos\varphi(t)\mathfrak{n}(t) + \sin\varphi(t)\mathfrak{b}(t)\right)\int_{w_0}^{w}\|c'(z)\|dz\right\} \qquad z, w \in \mathbb{C}$$

For $\varphi(t) \equiv \pi/2$ we obtain the classical formulation

$$\mathfrak{B}(S) = X(w) = \Re\left(c(w) - i\mathfrak{b}(t)\int_{w_0}^{w}\|c'(z)\|dz\right) \qquad z, w \in \mathbb{C}. \tag{12}$$

The Björling data in expression (12) reduces to $(c(t), \mathfrak{n}(t))$ and we write $S(t) = (c(t))$ since the normal vector and the principal normal to the curve coincide. When there are not way to confusion we speak about the "Björling transformation of $c(t)$" or simply "the Björling of $c(t)$" and we assume that $n(t) = \mathfrak{n}(t)$.

2.2.1 The Björling classes

We say that two Björling data $S(t)$ and $\hat{S}(t)$ are *Björling related* if they produce equivalent minimal surfaces. We will write $S \sim \hat{S}$ for related Borling's data. Equivalently, if the Björling data are given by the curves and their principal normals then we write $c \sim \hat{c}$.

The uniqueness of the solution implies that we can take two arbitrary geodesics $c, \bar{c} \subset X$ and its spherical images $n = N|_c$ and $n = N|_{\bar{c}}$ with regular parameterizations to produce the Björling data $S(t)$ and $\bar{S}(t)$. By construction $\mathfrak{B}(S)$ and $\mathfrak{B}(\bar{S})$ are equivalent surfaces and $S \sim \bar{S}$. In this way, we find families of infinitelly many related Björling data.

We consider viable strips as Björling data to have a parameterization defined in a maximal domain, which means in some punctured Riemann surface. With this condition, it is an excercise to proof the following

Lemma 2.1 $\sim$ *is an equivalence relation*

Example 1 The strips

$$S(t) = \{(t, 0, 0), (0, \cos(t), \sin(t))\}$$

and

$$\hat{S}(t) = \{(t, 0, 0), (0, \cosh(t), \sinh(t))\}$$

have the helicoid as common Björling transformation, therefore $S(t) \sim \hat{S}(t)$.

We can consider the classes of equivalence $[S]$ of all viable strips S such that $\mathfrak{B}(S) = X(w)$. We are interested in particular strips such that the Björling data reduce to planar curves.

2.3 Schwarz's reflections and symmetries

Schwarz discovered some interesting symmetry properties using expression (8). Such symmetries were used to construct a lot of minimal surfaces concatenating fundamental domains of minimal surfaces whose boundary is a composition of straight lines and/or plane geodesics. In order to glue two fundamental domains they must lie in the interior of a regular frame called a *Schwarzian chain* $\mathfrak{C}$. We use those symmetries for analyse the Björling transformation of symmetric supporting curves.

A symmetry A of a parametric minimal surface X induce an isometry $\alpha : \Omega \to \Omega$ such that $N \circ \alpha = \pm A \circ N$ where A is a rigid mouvement in $\mathbb{R}^3$. Since

the spherical image of X is invariant under translations, we are interested only in matrices $A \in O(3)$.

Let $\tau, \lambda : \Omega \to \Omega$ be functions given by

$$\begin{array}{rcl} \tau(w) & = & \bar{w} \\ \lambda(w) & = & iw, \qquad i = \sqrt{-1} \end{array}$$

and matrices $T, \Lambda \in O(3)$ given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

which span two representations of the diedral group D_4 in $\mathbb{C}^*$ and $GL_3(\mathbb{R})$ respectively. We have the identities

$$\tau^2 = \lambda^4 = \mathrm{Id}, \quad \lambda^{-1} = \tau\lambda\tau, \qquad T^2 = \Lambda^4 = \mathrm{Id}_4, \quad \Lambda^{-1} = T\Lambda T.$$

and in particular, τ is anticonformal and λ is conformal.

Considering the opposite orientation of the normal field in the solution of Björling's problem, Schwarz obtained the same minimal surface with the reflected domain $\bar{\Omega} = \{\bar{w} | w \in \Omega\}$. It has become his celebrated *reflection principle*.

Lemma 2.2 *Let $X : \Omega \to \mathbb{R}^3$ be a nonconstant minimal surface whose domain of definition Ω contains some interval I that lies on the real axis.*

- *ı) If the curve $c(u) = \{X(u) : u \in I\}$ is contained in some plane E, and if the surface X intersects E orthogonally at $c(u)$, then E is a plane of symmetry for X.*
- *ıı) If the image of $l(u) = \{X(u) : u \in I\}$ is contained in some line $\mathscr{L}$, then $\mathscr{L}$ is a line of symmetry of X.*

We assume that the line $\mathscr{L} \subset \tilde{X}$ belongs to the z-axis and the plane E is the xy-plane. Then *i)* corresponds to $X \circ \tau = T \circ X$ and *ii)* gives $X \circ \tau = -T \circ X$.

We have selected $\mathscr{L} \subset$ z-axis by convenience, in order that the spherical images of $c(u)$ and $l(u)$ concide in $\mathbb{S}^2$. In fact, they are projections of the same real curve $h : I \to \mathbb{C}^3$ with $h(u) = c(u) + il(u) \subset f(w)$. In this case $f : \Omega \to \mathbb{C}^3$ is the isotropic curve $f(w) = X(w) + iX^*(w)$. These relationships are contained in the next

Proposition 2.3 *Let $X : \Omega \to \mathbb{R}^3$ be a nonconstant minimal surface and assume that $X^* : \Omega \to \mathbb{R}^3$ is an adjoint minimal surface of X. Choose a smooth curve $\gamma : I \to \Omega$ with $\dot{\gamma}(t) \neq 0$ except for isolated points t_i in the interval I, and consider the curves $c(t) = X \circ \gamma(t)$ and $c^*(t) = X^* \circ \gamma(t)$. The following properties holds:*

(i) If c is a straight arc, then it is both a geodesic and an asymptotic line of X, and c^* is a planar geodesic of X^*. The curve c^* lies in some plane E and X^* intersects E orthogonally along c^*.

(ii) If c is a planar geodesic on X, then c^* is a straight arc (and hence a geodesic asymptotic line) on X^*.

Assume that $c(t) \subset X(w)$ is a geodesic contained in the XY-plane, then we have

$$(X + X^*)(\tau w) \quad = \quad T \circ (X - X^*)(w), \qquad w \in \Omega. \tag{13}$$

In other words $f(\tau w) = T \circ \overline{f(w)}$ where T acts on $\mathbb{C}^3$ by the diagonal action. This result comes from the holomorphic properties of f. The reader can see [2] for the proof of Lemma 2.2 and Proposition 2.3.

Definition 2.4 *Suppose that $c : I \to \mathbb{R}^2$ has a symmetry line $\mathscr{L} = \mathscr{L}(t)$ parameterized by $\mathscr{L}(t) = at + b$ with $a, b \in \mathbb{R}^3$ and $a \neq 0$. We say that c is a* perpendicular symmetric curve *with respect to $\mathscr{L}$ if there exist $t_0 \in I$ such that $c(t_0) \in \mathscr{L}$ and $\langle \dot{c}(t_0), a \rangle = 0$, We call the point $p = c(t_0)$ a* symmetry vertex *of c.*

We say that a perpendicular symmetric curve is non-degenerated *if its normal vector $n = \ddot{c}/\|\ddot{c}\|$ does not vanishes at its symmetry vertex.*

In this paper we are concerned with perpendicular non-degenerated symmetric curves. Non-degeneracy avoids umbilical points in the minimal surface at the symmetry vertex of $c(t)$. The reason is that umbilical points in minimal surfaces implies the vanishing of the principal curvatures κ_1 and κ_2 which are necessary in order to get perpendicular straight arcs. It is a consequence that at umbilical points a minimal surface is not conformal to its spherical image. Some examples of this failure are the high order element of the Enneper Family [2] or the high genus Costa surfaces [3].

Lemma 2.5 *Suppose that $c(t)$ is a perpendicular symmetric curve belonging to the XY-plane. Then*

$$\Lambda^2 T \circ X(w) = X(-\bar{w}) \tag{14}$$

Proof. This is immediate from the fact that $\bar{w} = \tau w$ and $-w = \lambda^2 w$ then $X(-\bar{w}) = X(\lambda^2 \tau w)$, and using Lemma 2.2 we obtain $X(\lambda^2 \tau w) = \Lambda^2 T \circ X(w)$. □

Lemma 2.6 *Let $c : I \to \mathbb{R}^3$ be a (non-degenerated) perpendicular symmetric curve and $X(w) = \mathfrak{B}(c)$ its Björling transformation. Then $\hat{c}(t) = X(\lambda t)$, $t \in I$ is a (non-degenerate) perpendicular symmetric curve.*

Proof. We suppose $c(t) \subset XY$-plane. Define $\hat{c}(t) = X(\lambda t)$ which is a well defined space curve. We must prove that $\hat{c}$ is a non-degenerated (planar) perpendicular symmetric curve. Using Lemma 2.5 we verify that $y(-\bar{w}) = -y(w)$. Then $y(it) = y(\lambda t) \equiv 0$ for all $t \in \mathbb{R}$. Writting $\hat{x}(t) = x(\lambda t)$ and $\hat{z}(t) = z(\lambda t)$ we obtain

$$\hat{c}(t) = (\hat{x}(t), 0, \hat{z}(t)) \,. \tag{15}$$

Which implies that $\hat{c}(t)$ is a planar curve. Applying $X(\tau w) = T \circ X(w)$ with $w = \lambda t$ we have

$$X(\tau \lambda t) = (x(\lambda t), 0, -z(\lambda t))$$

it means

$$\hat{c}(-t) = (\hat{x}(t), 0, -\hat{z}(t)) \,.$$

Then $\hat{c}(t)$ is symmetric with respect to the X-axis. Finally, its principal normal at the symmetry vertex does not vanish since $\hat{\mathfrak{n}}(0) = -\mathfrak{n}(0)$ and $c(t)$ is non-degenerated.

We conclude that $\hat{c}(t)$ is a non-degenerated perpendicular symmetric curve and $\hat{c}(t) \in [c(t)]$ by construction. $\square$

Definition 2.7 *Two perpendicular symmetric planar curves c and $\hat{c}$ are called* conjugated perpendicular geodesics under the Björling transformation *(or simply CPG), if for any parameterization of $c(t)$ such that $c(t) = X(t)$, for all $t \in I$ then $\hat{c}(t) = X(\lambda t)$ up to sign.*

In what follows we write only CPG to mean "the conjugated perpendicular geodesic curves under the Björling transformation".

We recall if $c(t)$ is an algebraic curve its analytic version $c(z)$ will be defined in some punctured Riemann surface and we can obtain global CPG curves.

Examples of CPG curves are the following:

- The circle and the catenary: both generate the Catenoid.
- The parabola and the cycloid: both generate the Catalan surface.
- The ellipse and a class of elliptical roulette: both generate the Elliptic catenoid studied in [6].
- The cubic $(t^2, t^3/3 - t)$ with itself: generate the Enneper surface.

The last example has the property that if $c(t) \subset XY$-plane then $\hat{c}(t) = \Lambda \circ c(t)$, $t \in I$ as defined above. We call them *self-CPG* curves. In fact, if $c : I \to \mathbb{R}^3$ is a self-CPG curve in the XY-plane, symmetric with respect to the X-axis and $X(w) = \mathfrak{B}(c)$ then $X(\lambda w) = \Lambda \cdot X(w)$.

In general, we consider the condition $X(\lambda t) = \Lambda \circ X(t)$ for $t \in I$ as the definition of the self-CPG curves.

Remark 1 *The CPG condition is not an equivalence relation. In [6] the author shows that the ellipse has two different CPGs, $c_1(t)$ and $c_2(t)$, which corresponds to the vertices of the ellipse but c_1 and c_2 are not CPG curves. The CPG condition is not transitive.*

Proposition 2.8 *Let $c, \hat{c} : I \to \mathbb{R}^3$ be two CPG (planar) curves and $X(w) = \mathfrak{B}(c)$ such that $c(t)$ is contained in the XY-plane and $\hat{c}(t) = X(it)$ contained in the XZ-plane. Then $\hat{c}(t) = \Lambda \cdot c(t)$ if and only if $X(t+it)$ and $X(t-it)$ are perpendicular straight lines in $X(w)$.*

Proof. We begin with the necessity. We suppose $c, \hat{c}$ are CPG and $X(t+it)$ and $X(t-it)$ are perpendicular straight arcs. Since $X(0)$ is not umbilical then any neigborhood of $X(0)$ is conformal to the disc $|z| < r$ for $z \in \Omega$ and $r > 0$ small. Since $c(t)$ is contained in the XY-plane and $\hat{c}(t)$ in the XZ-plane, then $X(t+it)$ belongs to $(0, y, y)$ and $X(t-it)$ belongs to $(0, y, -y)$.

Since $X(t+it)$ is a symmetry line every point in $c = (x, y, 0)$ is mapped under the symmetry to $\hat{c} = (-x, 0, y)$. It means that

$$\hat{c} = \Lambda T \cdot c. \tag{16}$$

The symmetry with respect to $X(t-it)$ implies that $c = (x, y, 0)$ is mapped to $\hat{c} = (-x, 0, -y)$. It means

$$\hat{c} = T\Lambda \cdot c. \tag{17}$$

Both curves are invariant under T therefore (16) and (17) implies $\hat{c} = \Lambda \cdot c$. Finally, $X : \Omega \to \mathbb{R}^3$ is conformal and an isometry then the holomorphic extension preserves distances from $c(t)$ to $\hat{c}(t)$, we conclude $c(t)$ is self-CPG.

Now the converse. We write $t' = (1-i)t$ and we have that $\lambda t' = \tau t'$. Since $c(t)$ is self-CPG we have

$$\Lambda X(t') = X(\lambda t') = X(\tau t') = TX(t'),$$

then $x(t-it) = -x(t-it)$ for all $t \in I$ and consequently $x(t-it) \equiv 0$. Additionally we obtain $y(t-it) = -z(t-it)$ for all $t \in I$ then $X(t-it)$ is contained in the line $(0, y, -y) \subset \mathbb{R}^3$.

On the other hand we write $t'' = (1+i)t$ and we consider the identity $\lambda\tau\lambda\tau = \mathrm{Id}$ to obtain $\lambda t'' = \lambda^2\tau\lambda\tau t'' = \lambda^2\tau t''$. The last equality comes from the invariance $(1+i)t = i \cdot \overline{(1+i)}t$. Then

$$\Lambda X(t'') = X(\lambda t'') = X(\lambda^2\tau t'') = \Lambda^2 TX(t''),$$

equivalently $X(t'') = \Lambda TX(t'')$. We obtain $x(t+it) \equiv 0$ and $y(t+it) = z(t+it)$, therefore $X(t+it)$ is contained in the line $(0, y, y) \in \mathbb{R}^3$. Perpendicularity is obvious. $\square$

Theorem 2.9 *Let $X : \Omega \to \mathbb{R}^3$ be a minimal surface such that $X(w) = \mathfrak{B}(c)$ for a self-CPG curve $c : I \to \mathbb{R}^3$, $I \subset \Omega$. Then the adjoint surface $X^* : \Omega \to \mathbb{R}^3$ is generated by another self-CPG curve $c^* : I' \to \mathbb{R}^3$ with $X^*(w) = \mathfrak{B}(c^*)$, for $I' \subset \Omega$.*

Proof. Since $c(t)$ is self-CPG then it belongs to some plane $E \subset \mathbb{R}^3$ which, as before, we assume $E = XY$-plane and it is symmetric with respect to the X-axis. From Proposition 2.8, $X(w)$ contains two perpendicular straight arcs $X(t+it)$ and $X(t-it)$ contained in the lines $(0, y, y)$ and $(0, y, -y)$ respectively.

Applying Proposition 2.3, the CPG curves $c(t)$ and $\hat{c}(t)$ are mapped to two perpendicular straight arcs $X^*(t)$ and $X^*(it)$ on the adjoint surface. Meanwhile the straight arcs $X(t+it)$ and $X(t-it)$ will be mapped to two planar geodesics $c^*(t) := X^*(t+it)$ and $\hat{c}^*(t) := X^*(t-it)$. By the self-$CPG$ definition, $c(t)$ is non-degenerated at $t = 0$, then $\mathfrak{n}(0) = -\hat{\mathfrak{n}}(0) \neq 0$ and the point $X(0)$ is not umbilical. These assures that the geodesics $c^*(t)$ and $\hat{c}^*(t)$ are perpendicular and therefore they are CPG.

From Proposition 2.8 we have $\hat{c}^*(t) = \Lambda \cdot c^*(t)$ or $\Lambda \cdot \hat{c}^*(t) = c^*(t)$ and consequently $c^*(t)$ is self-CPG. □

Corollary 2.10 *If $c : I \to \mathbb{R}^3$ is self-CPG then the isotropic curve $f(w) = X(w) + iX^*(w)$ where $X(w) = \mathfrak{B}(c)$ has a D_4 symmetry.*

Proof. It is enough to define $\Lambda f(w) = f(\lambda w)$ and $T\overline{f(w)} = f(\tau w)$ by diagonal action. □

Definition 2.11 *Let $X : \Omega \to \mathbb{R}^3$ be a minimal surface and $X^* : \Omega \to \mathbb{R}^3$ an adjoint surface to X. We say that X is* self-adjoint *if there exists an orthogonal matrix $R \in O(3)$ and an isometry $\rho : \Omega \to \Omega$ such that*

$$R \circ X^*(w) = X \circ \rho(w), \qquad w \in \Omega.$$

Corollary 2.12 *Every isotropic curve $f : \Omega \to \mathbb{C}^3$ whose components are self-adjoint minimal surface $RX^*(w) = X(\rho w)$ obtained by the Björling transformation of a self-CPG curve $c : I \to \mathbb{R}^3$ has a D_8 symmetry.*

Proof. This is immediate from the fact that the minimal surface $X(w) = \mathfrak{B}(c)$ for a self-CPG curve $c(t)$ has a D_4 symmetry. Applying Proposition 2.8, $X(w)$ posses two straight lines which are mapped to two geodesics in its adjoint surface $X^*(w)$. We define the complex matrix

$$R = \begin{pmatrix} i & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{18}$$

which maps the self-CPG curves to the perpendicular straight lines and viscev-ersa. Writing $\rho = \exp(\pi i/4)$ then we have

$$Rf(w) = f(\rho w), \qquad T\overline{f(w)} = f(\tau w)$$

which are the generators of the D_8 representation in $GL(3, \mathbb{C})$. Note that $\Lambda = R^2$ and $\lambda = \rho^2$. □

It is well-known that the Enneper surface and its adjoint are the same geometric object. In this way, it is a self-adjoint surface.

Another interesting example is the Costa surface $\mathscr{C}_1 : S_1 \to \mathbb{R}^3$, [3] where S_1 is the punctured Riemann surface associated to $w^2 = z(z^2 - 1)$ although the authors do not know a suitable parameterization of its self-CPG supporting curve.

3 Additional discussion

In this note we have characterized the adjoints to minimal surfaces which contains self-CPG curves. We can say that a minimal surface $X : \Omega \to \mathbb{R}^3$ which arise as the Björling transformation of a self-CPG curve $c : I \to \mathbb{R}^3$, has a D_4 symmetry. This symmetry is extended to the isotropic curve $f(\Omega) \subset \mathbb{C}^3$ since the adjoint surface X^* arises also as the Björling of another self-CPG curve. For the case of self-adjoints surfaces coming from self-CPG curves the D_4 symmetry is extended to a D_8 symmetry on the isotropic curve $f(w) = X(w) + iX^*(w)$ with generator (18).

There are other interesting examples which do not fall in the characterization given in this document. The family of algebraic curves

$$c_k(t) = \left\{ \left(\frac{2}{m} t^m, \frac{1}{2m-1} t^{2m-1} - t \right) | m = 4k - 2,\ k \in \mathbb{N} \right\}, \tag{19}$$

is a family of perpendicular symmetric curves whose Björling transformation $\mathfrak{B}(c_k)$ has a D_{2k+2} symmetry. The generator $\Lambda = \Lambda_k$ has the form

$$\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\frac{\pi}{2n} & -\sin\frac{\pi}{2n} \\ 0 & \sin\frac{\pi}{2n} & \cos\frac{\pi}{2n} \end{pmatrix}, \qquad n = 2k + 2.$$

If $k > 1$ we say that $c(t)$ is a *weak* CPG curve. In that case the sufficiency condition in Proposition 2.8 is not fulfilled.

The family (19) corresponds to the high order Enneper surfaces [2] and the same symmetries are shared by the high genus Costa surfaces [3]. In both cases the symmetry arises since the origin is an umbilical point.

We have several questions we are interested in to answer.

Question: What are the conditions for some curve $c : I \to \mathbb{R}^2$ to be self-CPG?

Question: Is it possible to deform the straight line to have a one parameter family of self-CPG curves?

Question: There are other self-adjoint surfaces which arises from self-CPG curves as in the case of the Enneper surface?

Question: Since there are a lot of examples of self-CPG curves whose Björling transformation gives embedded surfaces in $\mathbb{R}^3$, there exists an embedded self-adjoint surface in $\mathbb{R}^3$?

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